MATH 2020 Advanced Calculus II

Tutorial 5

Oct 8,10

1. Find the moments of inertia of the thin rectangular plate of constant density δ defined by $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 1$ with respect to the x-axis and y-axis.

   Solution.

$$\begin{aligned} I_x &= \int_0^2 \int_0^1 y^2 \delta dy dx \\ &= \frac{2\delta}{3} \end{aligned}$$

$$\begin{aligned} I_y &= \int_0^2 \int_0^1 x^2 \delta dy dx \\ &= \frac{8\delta}{3}. \end{aligned}$$

2. Find the centre of mass of the thin triangular plate of density $\delta = \delta(x, y) = x + 2y + 1$ bounded by $x = 0, y = 0$ and $x + y = 1$.

   Solution.

$$\begin{aligned} M &= \int_0^1 \int_0^{1-x} (x + 2y + 1) dy dx \\ &= \int_0^1 [(1 - x)(x + 1) + (1 - x)^2] dx \\ &= 1 \end{aligned}$$

$$\begin{aligned} M_x &= \int_0^1 \int_0^{1-x} x(x + 2y + 1) dy dx \\ &= \int_0^1 [(1 - x)x(x + 1) + x(1 - x)^2] dx \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} M_y &= \int_0^1 \int_0^{1-x} y(x + 2y + 1) dy dx \\ &= \int_0^1 \left[\frac{1}{2}(x + 1)(1 - x)^2 + \frac{2}{3}(1 - x)^3\right] dx \\ &= \frac{3}{8} \end{aligned}$$

$$\therefore (\bar{x}, \bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{1}{3}, \frac{3}{8}\right).$$

3. Let S be the solid bounded by $z = y^2, z = 1$ and $x = \pm 1$. Assume S has constant density. Find I_x, I_y, I_z and the centre of mass of S in terms of its mass M.

Solution. Let V be the volume of S.

$$\begin{aligned} V &= \int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} dzdydx \\ &= 2\int_{-1}^{1}(1-y^2)dy \\ &= \frac{8}{3} \end{aligned}$$

$$\begin{aligned} M_x &= \frac{M}{V}\cdot\int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} xdzdydx \\ &= 0 \end{aligned}$$

$$\begin{aligned} M_y &= \frac{M}{V}\cdot\int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} ydzdydx \\ &= \frac{2M}{V}\cdot\int_{-1}^{1} y(1-y^2)dy \\ &= 0 \end{aligned}$$

$$\begin{aligned} M_z &= \frac{M}{V}\cdot\int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} zdzdydx \\ &= \frac{2M}{V}\cdot\int_{-1}^{1} \frac{1}{2}(1-y^4)dy \\ &= \frac{8M}{5V} \end{aligned}$$

Let

$$\begin{aligned} X &:= \int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} x^2dzdydx \\ &= \frac{2}{3}\int_{-1}^{1}(1-y^2)dy \\ &= \frac{8}{9} \end{aligned}$$

$$\begin{aligned} Y &:= \int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} y^2dzdydx \\ &= 2\int_{-1}^{1} y^2(1-y^2)dy \\ &= \frac{8}{15} \end{aligned}$$

$$
\begin{aligned}
Z &:= \int_{-1}^{1}\int_{-1}^{1}\int_{y^2}^{1} z^2 dz dy dx \\
&= \frac{2}{3}\int_{-1}^{1}(1-y^6)dy \\
&= \frac{8}{7}
\end{aligned}
$$

Then

$$
(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_x}{M},\frac{M_y}{M},\frac{M_z}{M}\right) = \left(0,0,\frac{\frac{8}{5}}{\frac{8}{3}}\right) = \left(0,0,\frac{3}{5}\right),
$$

and

$$
\begin{aligned}
I_x &= (Y+Z)\frac{M}{V} = \left(\frac{8}{15}+\frac{8}{7}\right)\frac{3M}{8} = \frac{22}{35}M \\
I_y &= (X+Z)\frac{M}{V} = \left(\frac{8}{9}+\frac{8}{7}\right)\frac{3M}{8} = \frac{16}{21}M \\
I_z &= (X+Y)\frac{M}{V} = \left(\frac{8}{9}+\frac{8}{15}\right)\frac{3M}{8} = \frac{8}{15}M.
\end{aligned}
$$

4. (a) Consider the transformation

$$
\begin{cases} u &= x+y \\ v &= x+3y \end{cases}.
$$

Express x, y in terms of u, v, and compute $\dfrac{\partial(x,y)}{\partial(u,v)}$.

(b) Compute $\iint_R (x^2+4xy+3y^2)dydx$ where R is the region in the first quadrant bounded by the lines $y=1-x, y=2-x, y=-\frac{1}{3}x$ and $y=-\frac{1}{3}x+1$.

Solution.

(a) By linear algebra,

$$
\begin{cases} x &= \frac{3}{2}u - \frac{1}{2}v \\ y &= -\frac{1}{2}u + \frac{1}{2}v \end{cases}.
$$

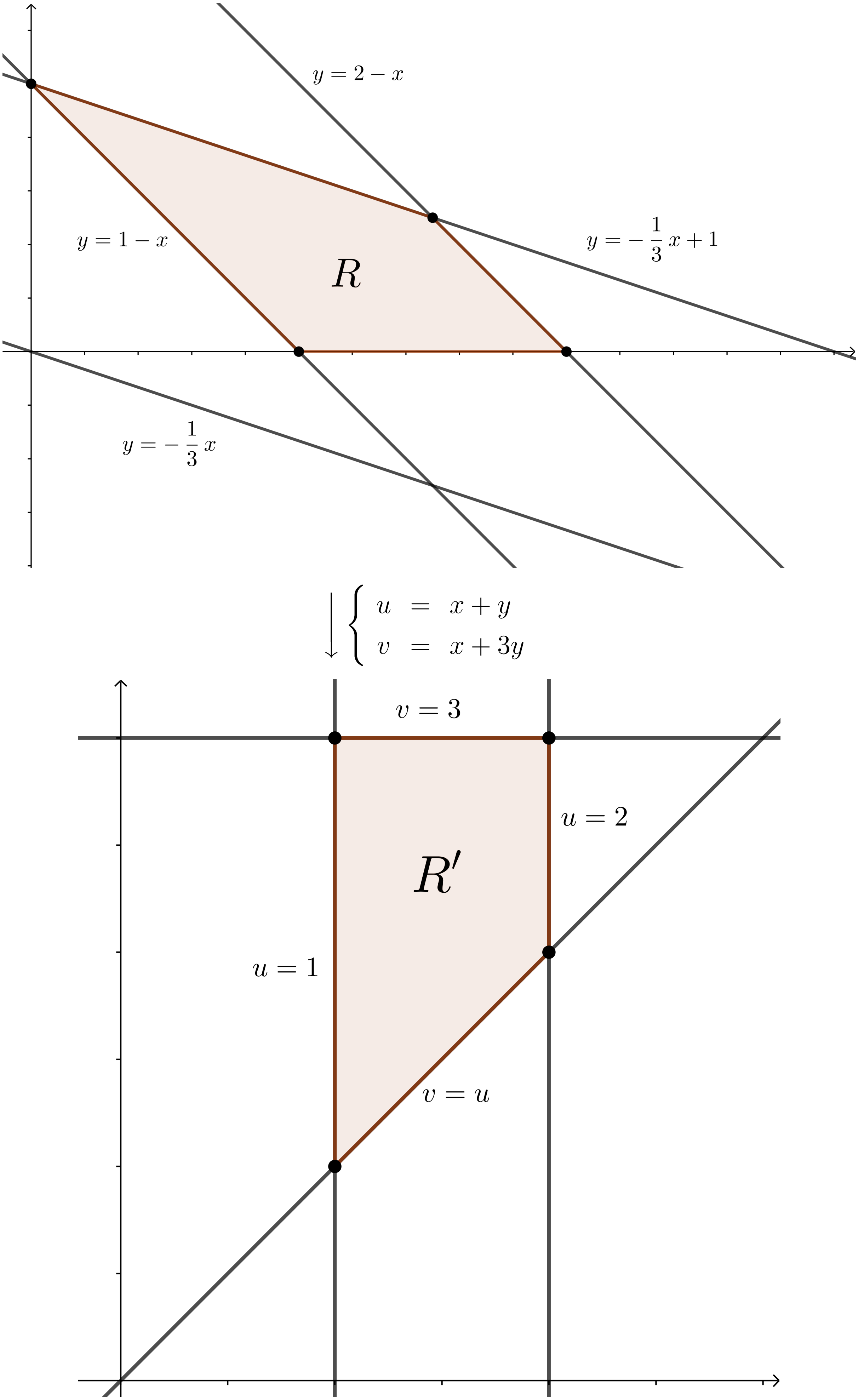
$$
\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.
$$

(b) Notice that $x^2+4xy+3y^2 = (x+y)(x+3y) = uv$, and

$$
\begin{matrix} y=1-x \iff u=1 \\ y=2-x \iff u=2 \end{matrix};\quad \begin{matrix} y=-\frac{1}{3}x \iff v=0 \\ y=-\frac{1}{3}x+1 \iff v=3 \end{matrix};
$$

$$
y=0 \iff -\frac{1}{2}u+\frac{1}{2}v = 0 \iff v=u.
$$

It follows that via the transformation described in (a), R is transformed into a right-angled trapezium bounded by $u=1, u=2, v=u$ and $v=3$. See the figure below. Hence the integral $= \int_1^2\int_u^3 uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dvdu = \frac{1}{2}\int_1^2\int_u^3 uvdvdu = \frac{39}{16}$.



Remark. The line $y = -\frac{1}{3}x$ is redundant for the region R, i.e. it does not contribute to its boundary.